

Rational Area Profiles of Regular Polygons Cut by Two Congruent Diagonals

Shoichi Inoue
shoichi31415@gmail.com

Keiichi Watanabe
keiichi@keiichi.wdev

Regular-polygon geometry is tightly linked to cyclotomic arithmetic: Poonen and Rubinstein's treatment of three-diagonal concurrence, for example, turns a geometric incidence condition into a short vanishing sum of roots of unity [PR98]. We prove an analogous rigidity result for areas. Two congruent crossing diagonals divide a regular n -gon into four regions. For the four regions cut out by the two congruent crossing diagonals V_0V_m and V_kV_{n-m+k} of a regular n -gon, we completely classify, for all parameters (n, k, m) , which sums of the normalized areas $a_0, \dots, a_3$ are rational. The classification has a sharp finite–infinite contrast: a_0 is rational in only five configurations, whereas the rational cases for a_2 and adjacent two-region sums form infinite families. Rationality is delicately sensitive to the parameters: for the configuration $(14, 3, 5)$ in Figure 1, no nontrivial subset sum is rational, while the neighboring cut $(14, 4, 5)$ gives $a_2 = 5/7$. The proof reduces each rationality condition to trigonometric relations at rational multiples of π and combines cyclotomic norm arguments with the classification theorems of Conway–Jones [CJ76] and Poonen–Rubinstein [PR98].

Geometric setting. By scale invariance, we may assume that P_n is a regular n -gon of circumradius 1. Let $V_0, V_1, \dots, V_{n-1}$ be its vertices in cyclic order. Assume

$$n \geq 5, \quad 2 \leq m < \frac{n}{2}, \quad 1 \leq k < m,$$

and consider the two chords

$$d_1 = V_0V_m, \quad d_2 = V_kV_{n-m+k}.$$

They have the same length and meet in the interior of P_n ; write $X = d_1 \cap d_2$. Since $0 < k < m < n - m + k < n$, the endpoints occur alternately on the boundary. Let Q_0 be the region bounded by V_0X , XV_k , and the boundary arc $V_0V_1 \cdots V_k$, and let Q_1, Q_2, Q_3 be the remaining regions in cyclic order. Reflection in the line through the center of P_n and the midpoint of the minor boundary arc from V_0 to V_k interchanges the two chords, and hence interchanges Q_1 and Q_3 . Therefore

$$a_1 = a_3, \quad a_i := \frac{\text{Area}(Q_i)}{\text{Area}(P_n)}.$$

Thus it suffices to classify one-region sums, adjacent two-region sums, and opposite two-region sums.

Theorem 1 (Rationality profile). *For every $I \subset \{0, 1, 2, 3\}$, the rationality of $A_I = \sum_{i \in I} a_i$ is determined by the following table. Define the following subsets of the parameter space, written in the order (n, k, m) :*

$$\begin{aligned} E_0 &= \{(6, 1, 2), (18, 3, 5), (18, 4, 6), (30, 3, 10), (30, 5, 12)\}, \\ E_2 &= \{(6, 1, 2), (9, 1, 3)\} \cup \left\{ \left(n, \frac{n}{2} - 3, \frac{n}{2} - 2 \right) : n \geq 8, n \text{ even} \right\}, \\ E_1 &= \{(6, 1, 2), (12, 1, 4)\}, \\ E_{\text{adj}} &= \left\{ (n, k, m) : n \text{ even}, m = \frac{n}{2} - 1, 1 \leq k < m \right\} \cup \{(12, 1, 3), (12, 2, 3)\}. \end{aligned}$$

Then

quantity	configurations for which it is rational
a_0	E_0
a_2	E_2
$a_1 = a_3$	E_1
$a_0 + a_1, a_1 + a_2, a_2 + a_3, a_3 + a_0$	E_{adj}
$a_0 + a_2, a_1 + a_3$	E_1 .

The empty sum, the full sum, and complements follow from $a_0 + a_1 + a_2 + a_3 = 1$. In particular, all four region areas are rational simultaneously if and only if $(n, k, m) = (6, 1, 2)$.

Area formulae. Put

$$\ell = n - 2m, \quad t = m - k, \quad \theta = \frac{\pi}{2n}.$$

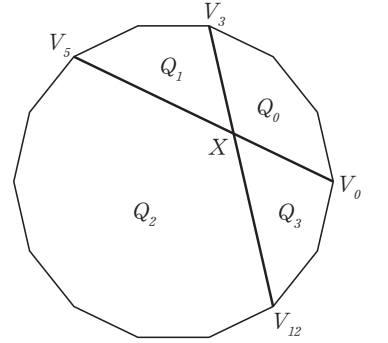


Figure 1: The configuration $(14, 3, 5)$: all nonempty proper region sums are irrational; the neighboring cut $(14, 4, 5)$ gives $a_2 = 5/7$.

The total area is $\text{Area}(P_n) = \frac{n}{2} \sin(2\pi/n)$. Decomposing the four regions into triangles gives

$$a_0 = \frac{k}{n} - \frac{2}{n}R, \quad R = \frac{\sin(k\pi/n) \sin(\ell\pi/2n)}{\sin(2\pi/n) \sin((2k + \ell)\pi/2n)}.$$

Similarly,

region sum	area expression	rationality target
a_2	$\frac{k + \ell}{n} + \frac{2}{n}S$	$S = \frac{\sin(\ell\theta) \cos((2t - \ell)\theta)}{\sin(4\theta) \cos(2t\theta)}$
$a_1 = a_3$	$\frac{t}{n} - \frac{1}{n}U$	$U = \frac{2 \sin^2(\ell\theta) \tan(2t\theta)}{\sin(4\theta)}$
$a_0 + a_1$	$\frac{m}{n} - \frac{1}{n}T$	$T = \frac{\sin(2m\pi/n)}{\sin(2\pi/n)}.$

Moreover $a_0 + a_2 = 1 - 2a_1$, so the opposite sums are governed by the same trigonometric quantity as a_1 . Since n, k, m, ℓ, t are integers, each area-rationality problem is equivalent to the rationality of the corresponding trigonometric quantity.

Proof sketch. The geometric reduction is direct, but the completeness of the classification is not. Each rationality target lies in a real cyclotomic field whose degree grows with n ; cyclotomic norm constraints first force a rational target to take one of a few possible values, and the surviving equalities become short cosine relations or three-sine product identities, to which global classification theorems apply.

The case of a_0 is the most rigid one. With $A = 2k$, $B = \ell$, $C = A + B$, the quantity R becomes

$$R = \frac{\sin(A\theta) \sin(B\theta)}{\sin(4\theta) \sin(C\theta)}.$$

If $R \in \mathbb{Q}$, a cyclotomic norm computation applied to $1 - \zeta^r$ in $\mathbb{Q}(\zeta)$, with $\zeta = e^{\pi i/n}$, shows that either $(n, k, \ell) = (6, 1, 2)$ and $R = 1/3$, or $R = 1$. The equation $R = 1$ is transformed by product-to-sum formulae into a short cosine relation

$$\cos(D\theta) - \cos(C\theta) - \cos((C - 4)\theta) + \cos((C + 4)\theta) = 0, \quad D = |A - B|.$$

Conway–Jones’ theorem on rational linear relations among at most four cosines of rational angles [CJ76] leaves only $(n, C, D) = (18, 14, 2)$ and $(30, 16, 4)$, giving the four remaining elements of E_0 .

For a_2 , the same norm argument restricts $S \in \mathbb{Q}$ to $S = 1$, except for the small case $(n, \ell, t) = (6, 2, 1)$, where $S = 2/3$. Solving $S = 1$ gives either $(n, \ell, t) = (9, 3, 2)$ or the structural family

$$\ell = 4, \quad t = 1, \quad n \geq 8 \text{ even},$$

which explains the infinite family in E_2 .

For $a_1 = a_3$, a preliminary norm argument shows that $U \in \mathbb{Q}$ forces $U = 1$, apart from the small case $(n, k, m) = (6, 1, 2)$. The equation $U = 1$, after replacing the cosine factor by a sine and clearing denominators, becomes a three-sine product equation of the type classified by Poonen–Rubinstein:

$$\sin(\pi u_1) \sin(\pi u_2) \sin(\pi u_3) = \sin(\pi v_1) \sin(\pi v_2) \sin(\pi v_3), \quad \sum u_i + \sum v_i = 1.$$

Their classification into trivial solutions, four one-parameter families, and sporadic solutions [PR98] yields exactly $(n, k, m) = (6, 1, 2)$ and $(12, 1, 4)$. Finally, adjacent two-region sums are just the cap cut off by a single diagonal; the quotient $T = \sin(2m\pi/n)/\sin(2\pi/n)$ is handled by a cyclotomic norm argument, giving E_{adj} .

Relation to previous work. Araki [Ara22] exhibited, up to relabeling, the instance $(n, k, m) = (18, 6, 7)$ with $a_2 = 2/3$; our theorem places it in the even- n family in E_2 and extends it to a complete classification of all normalized region sums. After the cyclotomic-norm reductions, the remaining equations for a_0 and a_2 are special cases of rational-angle sine-product equations classified by Myerson [Mye93]. For $a_1 = a_3$, we use the three-sine product classification of Poonen and Rubinstein [PR98], developed in their study of three-diagonal concurrence in regular polygons. Our contribution is the geometric reduction, the preceding norm restrictions, and the synthesis of these ingredients into a complete rationality profile for the present configuration.

References

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